Information Paths and the Determination of State Relations from Displacement Velocity Measurements of Elastic Rods

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Abstract

New methods are developed for determining the nonlinear compressional state relation between stress and strain for one dimensional elastic materials from measurements of displacement velocity. An iterative procedure is proposed when the early displacement velocities are provided at both ends of the sample. This proceedure is rapidly convergent and simple to implement in the weakly nonlinear regime. Non-iterative procedures are developed for displacement velocity data provided at an end and a location in the interior of the sample or for displacement velocity data provided at an end and at a fixed laboratory coordinate.

I-Introduction:

The relation between a seismic signal and the disturbance generated by a known source is determined by the state relation between stress and strain everywhere in the intervening medium together with conditions at boundaries. In geological applications, the mathematical models often employed to analyze the wave propagation dynamics between sources and receivers are hyperbolic systems of partial differential equations (e.g., first order conservation of mass and momentum, second order wave or fourth order biharmonic). These models, or principals derived from them (e.g., ray tracing) are often useful in analyses of inverse problems that arise in reflective seismology [1]. For example, a problem common to exploration, nonproliferation, and treaty verification is the determination of the location, geometry, and strength of sources and scatterers of seismic signals. In recent years, the importance of finite amplitude (i.e., nonlinear) effects has been recognized [2]-[9]. When effects due to nonlinearity are significant the precision with which the state relation is modeled will affect the accuracy of predictions and interpretions of data.

Motivated by the need to accurately model state relations, a different inverse problem is treated in this paper. An attempt is made to determine a state relation compatible with a known disturbance generated at a known location and the signal received. This infomation set is sometimes generated in laboratory studies of rock samples, and the methods proposed in this paper are motivated by resonant bar and nonlinear wave propagation experiments conducted at Los Alamos National Laboratory. A long thin right circular rod is made to oscillate at one end using a piezo-electric crystal and the induced motion of the free boundary at the far end and other positions is measured with accelerometers [10]. Recent attempts to infer a state relation for the sample in the weakly nonlinear regime employed an iterative proceedure using a Green's function technique for the one dimensional (axial) time-reduced wave equation to determine a polynomial approximation to the relation between stress and strain [11]. This method has not yet resulted in a reliable predictive capability for the frequency and amplitude scalings of the modes measured at the free end of the rod [12]. Modifications to the proceedure are being made, but a different method may be needed.

The need to predict and explain the experimental configuration described above is the motivation for the methods derived in this paper for determining state relations from velocity measurements in elastic rods. It is assumed that one-dimensional models provide an adequate description of these experiments, and the stresses applied at one end of a rod used to generate the needed data do not generate shocks in the interior of the sample A first order system for the density and displacement velocity is employed; conservation of mass and momentum. Two types of data are considered, and techniques are developed that allow the determination of a state relation from them. For data consisting of displacement velocity measurements at both ends of the bar, a "shooting" method is proposed. The trajectories of one family of "characteristics", or information paths, of the model are "aimed" in space-time from the driven end of the bar. Knowledge of the space-time direction of these paths at the driven end determines a state relation. The proposed shooting method uses the stress-free boundary condition at the free end and the driving velocity at the other. The "aiming" of the characteristics is "corrected" with successive iterations to account for the error in displacement velocity produced by the approximate state relation at the unstressed end. With each redirection of the space-time paths, a new and more accurate approximate state relation results. The method is mathematically straight forward and is rapidly convergent in the weakly nonlinear regime. It is also compatible with the possibility of stress-strain relations that are not analytic. For small stresses and strains the first approximation to the characteristic paths results in a very accurate approximation to the true state relation.

The method provides an exact state relation without iteration if the displacement velocity at a fixed Lagrangian coordinate inside the rod is given (e.g., via an imbedded diagnostic) rather than at a free end. Proceedures are also prescribed for inferring the exact state relation without iteration if data is taken at a fixed laboratory coordinate (i.e., a midpoint position of the rod) rather than at a free end. This will be the case if either the local velocity of the the rod is measured as it passes the fixed coordinate, or if the fractional mass of the rod on either side of the fixed coordinate is measured as a function of time. (The latter measurement could be made with the aid of an optical labeling of the unstressed rod.)

The outline of the paper is as follows: The model and notation are introduced and the inverse initial boundary value problem is formulated in the next section for data taken at an unstressed end. Section III is a treatment of important space-time properties of the information paths for the model. An essential relation between the forced motion at one end of a bar and the motion induced at its free end is derived in that section. The shooting method is described in Section IV for data taken at an unstressed end. Properties of the first approximation to the state relation obtained from the method are described in Section V and illustrative examples are given in Section VI. A method for determining exact state relations from midpoint data without iteration is given in Section VII. A summary of results is given in Section VIII. The Appendix is a brief treatment of effects of the convexity of the state relations on corrections to the approximations of state relations.

II-<u>Mathematical Model and an Inverse Problem</u>:

In this section longitudinal wave propagation is modelled in a cylindrically symmetric elastic bar of small radius. The elastic stress, σ, of the bar is assumed to be a function space and time only via strain. Plasticity and hysteresis effects are not considered. By "small radius" we mean that for a good description of the waves considered, radial effects may be neglected. If azimuthal symmetry is assumed, a one-dimensional model for compressional wave propagation is then given by the continuity and force balance equations for the density, $\rho(z,t)$, and axial displacement velocity, $v(z,t)$. (The laboratory axial position coordinate is z, and time is t.).

$$\begin{aligned} (\rho_t + v\rho_z) + \rho v_z &= 0 \\ \rho(v_t + vv_z) + \sigma_z &= 0 \end{aligned} \tag{1a}$$

Subscripts t or z indicate partial differentiation with respect to the subscripted variable with the other held constant.

It will be convenient to write eqs.(1) in Lagrangian coordinates. If at time t, the bar occupies the region

$$Z_o(t) < z < Z_M(t)$$

we define the Lagrangian mass and time coordinates for the bar to be

$$m(z,t) = \int_{Z_0(t)}^{z} \rho(z',t)dz'$$
$$T = t$$

$0 < m < M = \rho^*L$, where ρ^* and L are the uniform density and length of the unstressed bar, respectively.

These relations, together with eqs.(1a), imply

$$\begin{aligned} \rho_T + \rho^2 v_m &= 0 \\ \rho v_T + \rho\sigma_m &= 0 \end{aligned} \tag{1b}$$

Eqs.(1b) may be written in the form

$$\begin{aligned} \frac{1}{\rho^*}\left(1 - \frac{\rho^*}{\rho}\right)_T + v_m &= 0 \\ v_T + \sigma_m &= 0 \end{aligned} \tag{1c}$$

Subscsripts T or m indicate partial differentiation with respect to the subscripted variable with the other held constant. (Using these coordinates displacements may obtained by integrating the function $v(m,T)$ with respect to T.)

The quantity

$$\xi \equiv \left(1 - \frac{\rho^*}{\rho}\right)$$

is a measure of the strain. (If a coordinate in the unstressed bar, z*, is displaced to the position, z, conservation of mass requires $\rho^*dz^* = \rho dz$.) ξ, which is the negative of the relative extension [13], will be referred to as the strain parameter. The strain parameter is an increasing function of density, running from minus infinity to plus one. It takes the value zero in the absence of strain (i.e., when $\rho = \rho^*$). The state relation will be written as either $\sigma(\xi)$ or $\xi(\sigma)$. Both functions vanish when their arguments are zero, and both are monotonically increasing functions of their arguments.

The system of eqs.(1c) and the state relation are a hyperbolic system of first order partial differential equations for the displacement velocity, v, and stress, σ (or strain parameter, ξ). The local charateristic paths (or signal "speeds") in (m,T) space are given by [14].

$$\frac{dm}{dT} = \pm \sqrt{\rho^*\sigma'(\xi)}$$

Suppose now that the state relation of the bar, assumed to be initially unstressed and stationary, is not known a-priori, and one end, the one at m = 0, is oscillated or made to move with a known velocity, v(0,T). The forced motion induces a motion, v(M,T), at the other (unstressed) end, which is measured. The problem addressed in the next sections of the paper is to determine $\sigma(\xi)$ from the known time histories of the motion of the ends.

This inverse problem, as we have posed it, need not have a solution. The measured boundary function pairs [v(0,T), v(M,T)] may not be compatible with our modeling assumptions. Moreover, if a state relation does exist, it is not obvious that it is unique. Since only special function pairs, which are easily identified, correspond to a linear relation between σ and ξ, nonlinear wave propagation and the possibility of shocks is expected. For any given equation of state, shocks may be avoided by restricting the acceleration applied at m = 0. In this paper, the attempt to find a state relation will assume the absence of shocks. This assumption imposes a simple condition (eq.(6) below) that a boundary function pair must satisfy as a necessary condition for a solution (i.e., a state relation) to the inverse problem to exist. Assuming the data meets this condition, an iteration proceedure for the determination of the state relation is proposed and then applied to approximate the stress-strain relation for weak stress.

It will be convenient to rewrite eqs.(1c) in terms of dimensionless variables. The definitions of the dimensionless mass, μ, and time, τ, are

$$\mu \equiv m/M \; ; \; \tau \equiv T\sqrt{\rho^*\sigma'(0)}/M$$

The bar occupies $0<\mu<1$. The dimensionless displacement velocity, $w(\mu, \tau)$, and stress, $s(\mu, \tau)$, are

$$w \equiv v\sqrt{\rho^*/\sigma'(0)} \; ; \quad s(\xi) \equiv \sigma(\xi)/\sigma'(0)$$

Note that s'(0) = 1. Both independent and dependent dimensionless variables appear to contain an unknown parameter, $\sigma'(0)$. However, the parameter is given in terms of the measured arrival time, T*, of the first disturbance from m = 0 at m = M by

$$\frac{\sigma'(0)}{\rho^*} = \left(\frac{L}{T^*}\right)^2$$

This relation follows from the fact that $\sigma'(0)/\rho^*$ is the square of the sound speed for eqs.(1a) in regions of zero stress, and the stress is zero everywhere on the propagation front of the onset of the disturbance.

The definitions of the dimensionless variables and eqs.(1c) yield

$$\begin{aligned}[\xi(s)]_\tau + w_\mu &= 0 \\ w_\tau + s_\mu &= 0\end{aligned} \tag{1d}$$

In eqs.(1c) we have emphasized the assumption of a one-to-one relation between stress and strain by writing the strain parameter as a function of the dimensionless stress. For simplicity of exposition in the treatment that follows, we will refer to ξ, s and w as the strain, stress and displacement velocity, respectively.

The signal paths in (μ,τ) space for eqs.(1d) are given by

$$\frac{d\mu}{d\tau} = \pm\sqrt{\frac{ds}{d\xi}} \tag{2}$$

In particular, if the stress vanishes everywhere on a characteristic path, the path is a straight line of slope plus or minus one. In general, if the stress is constant on a characteristic, that characteristic is a straight line.

A signal path through a point P = (μ,τ) will be referred to as the $C^\pm$ characteristic (depending on the sign of $d\mu/d\tau$) through P, and will sometimes be written $C^\pm(P)$ or $C^\pm(\mu,\tau)$. The quantities

$$\chi^\pm \equiv w \pm \int_0^\xi \sqrt{\frac{ds(\tilde{\xi})}{d\tilde{\xi}}}\, d\tilde{\xi} = w \pm \int_0^s \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s}$$

satisfy

$$D_\tau^\pm \chi^\pm \equiv (\chi^\pm)_\tau \pm \sqrt{\frac{ds}{d\xi}}(\chi^\pm)_\mu = 0 \tag{3}$$

That is χ^+ and χ^- are constant on C^+ and C^- characteristics, respectively.

III-<u>Characteristics and the State Relation</u>:

Suppose that the bar is unstressed for $\tau < 0$, and an acceleration is applied to the bar at $\mu = 0$ when $\tau = 0$. The applied acceleration causes stress and displacement velocity waves to propagate in the bar which will be analysed using the C^+ and C^- characteristics. Analysis of the evolution and propagation of these waves is facilitated by partitioning (μ,τ) space into three regions as shown in Fig.1. The driven and stress-free ends of the bar are at $\mu=0$ and $\mu=1$, respectively. C^+ and C^- characteristics are shown as curves broken by + and - signs, respectively. (In the special case of a linear state relation, the signal speed is independent of strain, and all characteristic paths are straight lines with slope ± 1.) Region I consists of (μ,τ) coordinates that have not received information that a stress was applied at $\mu = 0$. Since both the displacement velocity and stress vanish if the bar is initially at rest and unstressed, the information paths in region I are straight lines with slopes ± 1. Region II consists of those (μ,τ) coordinates that have received information that a disturbance has been generated at $\mu = 0$ but have not felt the effects of reflections from the free end of the bar. In the treatment that follows it is shown that the C^+ characteristics in region II are straight lines whose slopes depend on the constant values of stress and strain that are carried from the driven end of the bar on each of these information paths. The displacement velocity is also a constant on these paths. C^- characteristics in region II are generally curvilinear since stress and strain vary along each such path. Information reflected from the free end is felt everywhere in region III. In general, both C^+ and C^- characteristics are curvilinear in this region. The C^+ characteristic common to regions I and II is the $C^+(0,0)$ characteristic, and the C^- characteristic common to regions II and III is the $C^-(1,1)$ characteristic. At the free end, $\mu=1$, the stress is zero, so all the local slopes of the characteristic paths there are ± 1.

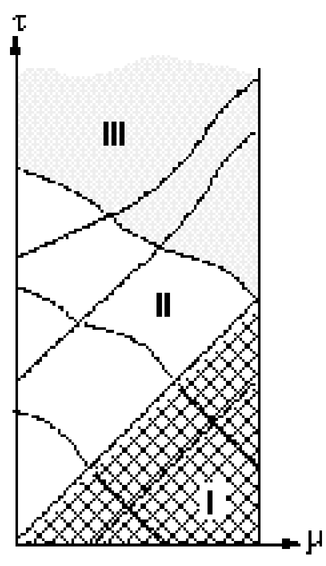

Figure 1 Illustrating regions I-III in the (μ,τ) coordinate system.

Suppose that a C^+ characteristic starting at $\tau = 0$ intersects a C^- characteristic starting at $\tau = 0$ (see Fig.1). Since both the displacement velocity, w, and the stress, s, are initially zero in the bar, and the quantities χ^+ and χ^- are constant on C^+ and C^- characteristics, respectively,

$$0 = w + \int_0^s \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s}$$

$$0 = w - \int_0^s \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s}$$

at the point of intersection. Consquently, both the displacement velocity and stress vanish. The same result is obtained if the C^+ characteristic starting at $\tau = 0$ intersects a C^- characteristic starting at $\mu = 1$ and a time τ prior to initial motion of the unstressed end. All characteristics in this region of (μ,τ) space have slope plus or minus one. The region is bounded by the line $\tau = \mu$ and is "region I" of Fig.1.

Now suppose that a C^+ characteristic statrting at $\mu = 0$ (i.e., the driven end) and positive τ, intersects a C^- characteristic starting in region I (see Fig.1). Since the quantities χ^+ and χ^- are constant on C^+ and C^- characteristics, respectively, at the point of intersection,

$$w + \int_0^s \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s} = w(0,\tau) + \int_0^{s(0,\tau)} \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s}$$

$$w - \int_0^s \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s} = 0 \tag{4}$$

In particular, for those pairs $(0,\tau)$ that lie on C^- characteristics starting in region I

$$w(0,\tau) = \int_0^{s(0,\tau)} \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s} = \int_0^{\xi(0,\tau)} \sqrt{\frac{ds(\tilde{\xi})}{d\tilde{\xi}}}\, d\tilde{\xi} \tag{5a}$$

Consequently, on information paths starting at these pairs

$$w + \int_0^s \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s} = 2w(0,\tau) \tag{5b}$$

Since the stress vanishes at $\mu = 1$, when one of these C^+ paths, $C^+(0,\hat{\tau})$, arrives at $\mu = 1$ at $\tau = \tau^+(\hat{\tau})$

$$w(1,\tau^+(\hat{\tau})) = 2w(0,\hat{\tau}) \tag{6}$$

In general, this relation between displacement velocities at different ends of the bar holds only on those C^+ characteristics starting from coordinates $(0,\tau)$ that also lie on C^- characteristics originating in region I. These C^+ characteristics are generated at $\mu = 0$ prior to the arrival of reflections from the unstressed end of the bar. They correspond to information propagated from the earlier motions of the driven end. In the absence of shocks, the speeds at both ends of these C$^+$ characteristics have the same sign. At the earliest positive times, therefore, so do the accelerations. (For linear state relations, Eq.(6) follows from a method of images [15] in which an equal and opposite wave is launched to satisfy the boundary condition at the free end. Since the wave propagation is linear, a pulse traverses the length of the rod undistorted and the velocities are superposed at the free end.) Along the intersection of such a C$^+$ characteristic with C$^-$ characteristics originating in region I, eqs.(4) yield

$$w = \int_0^s \sqrt{\frac{d\xi(\tilde{s})}{d\tilde{s}}}\, d\tilde{s} = w(0,\tau) \tag{7}$$

That is, the stress is constant, so the C$^+$ characteristic is a straight line in this region of (μ,τ) space. This is region II of Fig.1. In the terminology of hydrodynamics, the solutions in region II are often referred to as "simple waves" [16].

Eq.(6) provides a diagnostic for testing the validity of the one dimensional model. Suppose the driven end of the bar is made to oscillate. If peaks and/or troughs of the driven oscillation occur prior to the arrival of reflections from the free end, the information generated at those peaks and/or troughs will be transmitted on C$^+$ characteristics through region II and arrive at the unstressed end of the bar. At that end, the measured displacement velocity peaks and/or troughs should be sequencially twice those imposed at the driven end if the model and assumptions are valid.

In region II, a parametric representation may be obtained for the C$^-$ characteristics. The C$^+$ characteristic through $(0,\hat{\tau})$ is a straight line

$$\mu = (\tau-\hat{\tau})\sqrt{\frac{ds}{d\xi}(0,\hat{\tau})} \tag{8}$$

In region II, a C$^-$ characteristic crosses these staight line paths, each of which is parameterized by $\hat{\tau}$. Using eq.(8) in eq.(2) for the C$^-$ characteristic yields integrable first order ordinary differential equations for $\mu(\hat{\tau})$ and $\tau(\hat{\tau})$. For the C$^-$ characteristic passing through the point $(\mu(0),\mu(0)) = (\tau(0),\tau(0))$ on the $C^+(0,0)$ characteristic

$$\mu\left[\frac{ds}{d\xi}(0,\hat{\tau})\right]^{-1/4} - \mu(0) = (\tau-\hat{\tau})\left[\frac{ds}{d\xi}(0,\hat{\tau})\right]^{1/4} - \tau(0)$$
$$= -\frac{1}{2}\int_0^{\hat{\tau}} \left[\frac{ds}{d\xi}(0,\tilde{\tau})\right]^{1/4} d\tilde{\tau} \tag{9}$$

The, $C^+(0,0)$ and $C^-(1,1)$ characteristics and the line $\mu = 0$ are the boundary of region II. The extent of this region depends on the state relation, $s(\xi)$.

The region covered by C$^+$ characteristics continuing from region II to the unstressed end, $\mu = 1$, is part of region III (see Fig.1).

IV-<u>An Iteration Scheme for s(ξ)</u>:

Suppose that the slopes of the $C^+(0,\hat{\tau})$ characteristics are given in region II. Then eq.(5a) yields

$$\sqrt{\frac{ds}{d\xi}(0,\hat{\tau})}\,\frac{d\xi(0,\hat{\tau})}{d\hat{\tau}} = \frac{dw(0,\hat{\tau})}{d\hat{\tau}} \tag{10}$$

The square root on the left-hand-side is just $d\mu/d\tau$ on the $C^+(0,\hat{\tau})$ characteristic. Since the time derivative of $w(0,\tau)$ is known, eq.(10) provides a functional relation between τ and ξ. The state relation associated with the given slopes may then be determined from eq.(5a) by differentiation

$$\frac{ds(\xi)}{d\xi} = \left[\frac{dw(0,\hat{\tau}(\xi))}{d\xi}\right]^2$$

$$s(\xi) = \int_0^{\xi} \left[\frac{dw(0,\hat{\tau}(\tilde{\xi}))}{d\tilde{\xi}}\right]^2 d\tilde{\xi} \tag{11}$$

Eqs.(10) and (11) provide the basis of an iteration scheme for obtaining an approximate state relation.

The scheme is a "shooting" approximation for the C^+ characteristics. It depends on the fact that a relationship exists, independent of the particular state relation, between the displacement velocities at the ends of $C^+(0,\hat{\tau})$ characteristics, provided $\hat{\tau}$ is sufficiently small. The relationship is given by eq.(6). The graphical determination of $\hat{\tau}(\tau^+)$ and $\tau^+(\hat{\tau})$. of eq.(6) for end point displacement velocity data $w(0,\tau)$ and $w(1,\tau)$ is illustrated in Fig.2. Two graphs of hypothetical velocity data are shown. It is assumed that $(\mu,\tau)=(0,\Delta\tau)$ is on the boundary of region II of Fig.1. Displacement velocity propagated on a C^+ characteristic through region II to the free end arrives at the free end with a value twice that in region II (see eq.(6)). Consequently, the functions $\hat{\tau}(\tau^+)$ and $\tau^+(\hat{\tau})$ may be constructed from these graphs via sequential association of times which are such that $2w(0,\hat{\tau}) = w(1,\tau^+)$. For the labelled times, $\tau^+(0) = 1$, $\hat{\tau}(1) = 0$ (see $C^+(0,0)$ characteristic of Fig.1) and $\tau^+(\hat{\tau}_i) = \tau_i^+$, $\hat{\tau}(\tau_i^+) = \hat{\tau}_i$ for i = 1,2,.....,6. Given the functions $w(0,\tau)$ and $w(1,\tau)$, and assuming neither is constant on a time interval of positive measure, eq.(6) allows an identification of the endpoints in (μ,τ) space of $C^+(0,\hat{\tau})$ characteristics. That is, the function $\tau^+(\hat{\tau})$ is known.

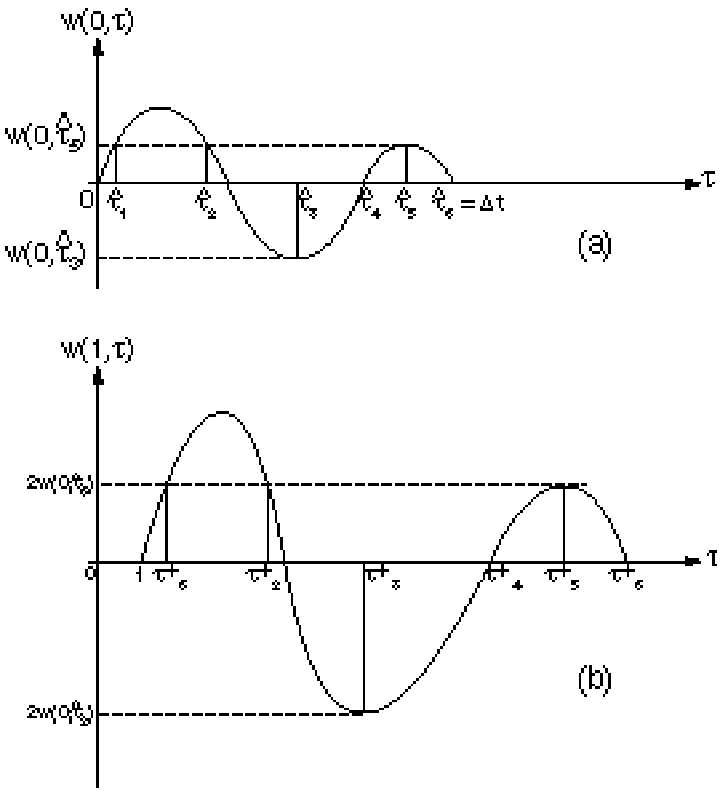

Figure 2 Construction of the functions $\hat{\tau}(\tau^+)$ and $\tau^+(\hat{\tau})$ for end point velocity data (a) $w(0,\tau)$ and (b) $w(1,\tau)$. Graphs of hypothetical velocity data are shown.

Assume, therefore, a pair of boundary data functions for the displacement velocity, w, is given, and that the function $\tau^+(\tau)$ has been determined for the pair. Since the $C^+(0,\hat{\tau})$ characteristics are straight lines. The slope of the straight line joining the points $(0,\hat{\tau})$ and $(1,\tau^+(\hat{\tau}))$ provides an approximation to the slope of the $C^+(0,\hat{\tau})$ characteristic in region II that connects these points if the path length in region III is not too large. The slope of the approximation at $(0,\hat{\tau})$ (actually everywhere in region II) is

$$\frac{d\tau}{d\mu} = \sqrt{\frac{d\xi}{ds}(0,\hat{\tau})} = \tau^+(\hat{\tau}) - \hat{\tau} \tag{12}$$

From eq.(10), the relationship between the time and the strain at $\mu = 0$ implied by this "approximation" of the C^+ characteristics is

$$\frac{d\xi(0,\hat{\tau})}{d\hat{\tau}} = \frac{dw(0,\hat{\tau})}{d\hat{\tau}}\left[\tau^+(\hat{\tau}) - \hat{\tau}\right] \tag{13}$$

Using this relationship in eq.(11) provides the state relation associated with this first approximation of the C^+ characteristics.This "First Approximate State Relation", or FAStR, may then be substituted in eq.(9) to determine the C^- characteristics and extent of the region II associated with this state relation. Unless the state relation is linear (i.e,.unless $\tau^+(\hat{\tau}) - \hat{\tau} = 1$), continuation of a $C^+(0,\hat{\tau})$ characteristic from this region II into the adjoining region III will not result in linear paths, and the coordinate $(1,\tau^+(\hat{\tau}))$ will be "missed". "Next approximations" to the correct $C^+(0,\hat{\tau})$ characteristics may be obtained by adjusting the slopes in region II to correct for the direction of the error at $\mu = 1$ incurred by previous approximations. In general, determination of the continuation of the $C^+(0,\hat{\tau})$ characteristics into the region III of an approximate state relation (with $w(1,\tau)$ to be determined compatibly with the approximate stress function) involves the solution of the resulting nonlinear wave equations in the associated region III with data given on the $C^-(1,1)$ characteristic.

For sufficiently small times, $\hat{\tau}$, this problem may be finessed by observing that the path length of the $C^+(0,\hat{\tau})$ characteristics in region III will be small. Estimates of the paths in region III using convenient functions may be attempted. For example, using a parabolic approximation to the path and matching the slope of a $C^+(0,\hat{\tau})$ characteristic to its value at the $C^-(1,1)$ characteristic boundary of region II while forcing the slope at the unstressed end $\mu = 1$ to be equal to one preserves the most important physical properties of the solution to the nonlinear equations in region III (continuity of stress and strain across the $C^-(1,1)$ characteristic and the stress-free character of the end at $\mu = 1$) and provides an accurate approximation to the $C^+(0,\hat{\tau})$ characteristic path error at $\mu = 1$ for small values of $\hat{\tau}$. If $\delta\tau^+(\hat{\tau})$ is the difference in the arrival time of the parabolic fit to an approximation and $\tau^+(\hat{\tau})$

$$\delta\tau^+(\hat{\tau}) = \frac{1}{2}\left[1 - \sqrt{\frac{d\xi}{ds}(0,\hat{\tau})}\right]\left(1 - \mu_\partial(\hat{\tau})\right) + \left[\sqrt{\frac{d\xi}{ds}(0,\hat{\tau})} - \left(\tau^+(\hat{\tau}) - \hat{\tau}\right)\right] \tag{14}$$

where $\mu_\partial(\hat{\tau})$ is the μ coordinate of the intersection of the $C^+(0,\hat{\tau})$ and $C^-(1,1)$ characteristics. For the FAStR, the second square bracket on the right hand side of eq.(14) vanishes. For small values of $\hat{\tau}$, both factors contributing to $\delta\tau^+(\hat{\tau})$ for the FAStR are small, suggesting the FAStR should be a very good approximation to the true state relation for small strains.

V-Properties of the First Approximate State Relation:

Expressions for the FAStR will now be obtained for small stresses and strains. For the FAStR,

$$\frac{ds}{d\xi} = \frac{1}{\left[\tau^+(\hat{\tau}) - \hat{\tau}\right]^2} \tag{15}$$

and eq.(10) takes the form

$$\frac{d\xi}{d\hat{\tau}}(0,\hat{\tau}) = \left[\tau^+(\hat{\tau}) - \hat{\tau}\right]\frac{dw(0,\hat{\tau})}{d\hat{\tau}} \tag{16}$$

Differentiating eq.(15) with respect to ξ, and using eq.(16), yields

$$\frac{d^2s}{d\xi^2} = - \frac{4}{\left[\tau^+(\hat{\tau}) - \hat{\tau}\right]^4 \frac{d\left[2w(0,\hat{\tau})\right]}{d\hat{\tau}}} \left(\frac{d\tau^+(\hat{\tau})}{d\hat{\tau}} - 1 \right) \tag{17}$$

The relation between τ^+ and $\hat{\tau}$ in eq.(6) implies

$$\frac{dw(1,\tau^+(\hat{\tau}))}{d\tau^+} \frac{d\tau^+(\hat{\tau})}{d\hat{\tau}} = \frac{d\left[2w(0,\hat{\tau})\right]}{d\hat{\tau}} \tag{18}$$

As a result, for earlier motions of the ends of the bar, derivatives of τ^+ with respect to $\hat{\tau}$ can be expressed in terms of derivatives of the measured velocities. Using eq.(18) in eq.(17) yields

$$\frac{d^2s}{d\xi^2} = \frac{4}{\left[\tau^+(\hat{\tau}) - \hat{\tau}\right]^4} \left(\frac{1}{\frac{d\left[2w(0,\hat{\tau})\right]}{d\hat{\tau}}} - \frac{1}{\frac{d\left[w(1,\tau^+)\right]}{d\tau^+}} \right) \tag{19}$$

In the absence of shocks, the earliest accelerations at the ends determine the convexity, at small strains, of the state relation. Similarly, the convexity of the state relation determines the early response of the free end to an acceleration of the other end.

Differentiation of eq.(19), and using eq.(18) as before, yields an expression for the third derivative of the stress with respect to strain for the FAStR.

$$\frac{d^3s}{d\xi^3} = 2\left[\tau^+(\hat{\tau}) - \hat{\tau}\right]^2 \left(\frac{d^2s}{d\xi^2} \right)^2 + \frac{4}{\left[\tau^+(\hat{\tau}) - \hat{\tau}\right]^5} \left\{ \frac{d}{d\hat{\tau}} \left(\frac{1}{\frac{d\left[2w(0,\hat{\tau})\right]}{d\hat{\tau}}} \right)^2 - \frac{d}{d\tau^+} \left(\frac{1}{\frac{d\left[w(1,\tau^+)\right]}{d\tau^+}} \right)^2 \right\} \tag{20}$$

If the indictated derivatives of the displacement velocity functions exist at $\hat{\tau} = 0$ and $\tau^+ = 1$ the expressions in eqs.(15), (19), and (20) provide the coefficients for the first three terms in a cubic approximation, or "Taylor" series expansion, of $s(\xi)$. (In general, one needs to know the first n derivatives of the velocity pair to determine the (n+1)st derivative of the state relation. If the velocity pair has singular behavior at $\hat{\tau} = 0$ and/or $\tau^+ = 1$ derivatives of $s(\xi)$ may become unbounded and approximation at small strain by polynomials beyond a certain order must be abandoned). Indicating differentiation by primes, the expansion up to and including the cubic term is then

$$s(\xi) = \xi + \frac{1}{2}s''(0)\xi^2 + \frac{1}{6}s'''(0)\xi^3 \tag{21}$$

If the second derivatives of the velocity functions vanish at $\hat{\tau} = 0$ and $\tau^+ = 1$ (e.g., $w(0,\tau) = A\sin\omega\tau$ for $\tau > 0$ and $w(1,\tau) = B\sin\Omega(\tau-1)$ for $(\tau - 1) > 0$) the third derivative at $\xi = 0$ is just twice the square of the second and eq.(21) becomes

$$s(\xi) = \xi + s''(0)\left(\frac{1}{2}\xi^2 + s''(0)\frac{1}{3}\xi^3\right) \tag{22}$$

A special case of interest arises when a simple sinusoidal oscillation is used to drive one end of the bar. Nonlinear coupling results in a spectrum of frequencies in the motion of the unstressed end. While eq.(22) results whenever $w(0,\tau)$ and $w(1,\tau)$ are finite sums of sine functions of τ and τ-1, respectively, its validity if either is a sum of infinitely many functions will depend, in general, on how rapidly the coefficients of the series approach zero. Truncation of a spectral representation must be done carefully in any effort directed at obtaining polynomial approximations to the state relation in the weakly nonlinear regime.

VI-<u>An Example</u>:

In practice it may not be possible to obtain simple mathematical relations between the variables τ^+ and $\hat{\tau}$. Tabulations are probably the best that can be expected. Even if analytical representations of the functional relation between the two are obtained, integrations required for implementation of the shooting method will probably require numerical procedures. In this section we illustrate the procedure for determining the FAStR for a deceptively simple displacement velocity pair. Both $w(0,\tau)$ and $w(1,\tau)$ are linear in time after $\tau = 0$ and $\tau = 1$, respectively. For displacement velocity pairs with inital accelerations different than zero, the linear approximation to the early applied velocity and free end response determines the convexity of the state relation at small strain. This first nonlinear term in the state relation dominates convergence and divergence of small amplitude signals propagating from the driven end to the unstressed end of the bar.

The FAStR is determined for a bar with applied and free end displacement velocity functions pairs $w(0,\tau) = \tau\alpha$ for τ non-negative and $w(1,\tau) = (\tau-1)\beta$ for $\tau-1$ non-negative. Both α and β are nonzero constants. It will be assumed that the velocity pair is consistent with the absence of shocks, so eq.(6) is valid for the times of interest and $\alpha\beta >$ 0.

Since no shocks form as a result of the applied acceleration, α, eq.(19) - (21) imply that for small strains

$$s(\xi) = \xi + 4\left(\frac{1}{2\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{2}\xi^2 + 4\left(\frac{1}{2\alpha} - \frac{1}{\beta}\right)\frac{1}{3}\xi^3\right) \tag{23a}$$

In fact, for this special velocity pair, all of the integrations required to find the FAStR can be effected analytically to obtain

$$s(\xi) = \frac{1}{4\left(\frac{1}{2\alpha} - \frac{1}{\beta}\right)}\ln\left(1 - 4\left(\frac{1}{2\alpha} - \frac{1}{\beta}\right)\xi\right) \tag{23b}$$

The cubic given by eq.(23a) is the first few terms in the Taylor series expansion of this complete relation. Note that even though the displacement velocity pair is linear in time the state relation is generally nonlinear. In the exceptional case, $2\alpha = \beta$, the FAStR, $s(\xi) = \xi$, is the exact linear state relation over the range of ξ at $\mu = 0$ in region II.

Having determined the state relation for the bar, suppose the end at $\mu = 0$ is later driven with velocity $w(0,\tau) = \tau\alpha'$, where α' a nonzero constant possibly different from α.

Since the convexity of the state relation is known, if there are no shocks the initial acceleration of the free end, β', must satisfy

$$\left(\frac{1}{2\alpha'} - \frac{1}{\beta'}\right) = \left(\frac{1}{2\alpha} - \frac{1}{\beta}\right) \tag{24a}$$

or

$$\frac{2\alpha'}{\beta'} = 1 - \frac{\alpha'}{\alpha}\left(1 - \frac{2\alpha}{\beta}\right) \tag{24b}$$

Since the left hand side must be positive, there is a constraint on α' for the applicability of the convexity relation in determining the response of the free end:

$$1 > \frac{\alpha'}{\alpha}\left(1 - \frac{2\alpha}{\beta}\right) \tag{24c}$$

If the constraint is not satisfied, application of the acceleration α' results in shocks and eq.(6) is not valid. This is easily seen to be the case using eq.(7) with $w(0,\tau) = \tau\alpha'$. To lowest order, $\xi = \tau\alpha'$ at $\mu = 0$. Using eq.(23a) and assuming small, nonnegative values of $\hat{\tau}$, the $C^{+}(0,\hat{\tau})$ characteristics in region II are given by

$$\tau - \hat{\tau} = \left[1 - \frac{\alpha'}{\alpha}\left(1 - \frac{2\alpha}{\beta}\right)\hat{\tau}\right]\mu$$

These characteristics intersect the $C^{+}(0,0)$ characteristic, $\tau=\mu$, at the mass coordinate μ given by

$$1 = \frac{\alpha'}{\alpha}\left(1 - \frac{2\alpha}{\beta}\right)\mu$$

Since, by hypothesis, the constraint relation, eq.(24c), is violated, the coefficient of μ is greater than or equal to one and the intersection occurs at values of μ between zero and one (i.e., within the bar), or at the free end, and shocks form.

If for given values of α and β both $|\alpha'| > 0$ and $-|\alpha'|$ satisfy the constraint relation, eq.(24b) yields

$$|\beta'| = \frac{|2\alpha'|}{\left|1 - 2\alpha'\left(\frac{1}{2\alpha} - \frac{1}{\beta}\right)\right|}$$

Consequently, $|\beta'|$ is larger if the signs of α' and the convexity of $s(\xi)$ are the same than if they are opposed. Thus, if the state relation has positive convexity, a push produces a more rapid acceleration at the free end that would a pull of the same initial acceleration, whereas the reverse would be true if the convexity of the state relation is negative.

VII- Exact State Relations from Measurements at a Fixed Laboratory Coordinate:

If data on the motion of the the elastic rod is provided at a fixed point in space (i.e., an Eulerian coordinate), then the results of Sections III and IV provide a methodology for determining the exact state relation without iteration. In terms of the Lagrangian coordinate of the rod, the fixed laboratory coordinate is x(m,T) = constant. At this fixed Eulerian coordinate, the Lagrangian mass coordinate, m, is a function of time, T. The differential relation between the two is obtained by differentiating the constant.

$$\begin{aligned} 0 &= x_m dm + x_T dT = \frac{1}{\rho} dm + v dT \\ 0 &= -(1-\xi) d\mu + w d\tau \end{aligned} \tag{25}$$

If the coordinate pair $(\mu(\tau),\tau)$ is in region II, the C^+ characteristic through $(\mu(\tau),\tau)$ originating at $(0,\hat{\tau}(\tau))$ is a straight line whose slope is given by

$$\sqrt{\frac{d\xi}{ds}(0,\hat{\tau}(\tau))} = \frac{\tau - \hat{\tau}(\tau)}{\mu(\tau)} \tag{26}$$

Since $w(\mu,\tau)$ and $\xi(\mu,\tau)$ are constant on a C^+ characteristic in region II, the analogue of eq.(6) that will used to determine $\hat{\tau}(\tau)$ is

$$w(0,\hat{\tau}(\tau)) = w(\mu(\tau),\tau) \tag{27}$$

Once the curve $(\mu(\tau),\tau)$ corresponding to the constant Eurlerian coordinate and the function $\hat{\tau}(\tau)$ are determined, eq.(26) provides the the exact slope of the C^+ characteristics at $\mu = 0$. It was shown that this is all that is needed to determine the state relation $s(\xi)$ over the range of ξ at the boundary of region II at $\mu = 0$. It will be shown that if (i) $\mu(\tau)$ is given (i.e., the rod is "labeled") then $\hat{\tau}(\tau)$ is determined by a nonlinear first order ordinary differential equation, or (ii) $w(\mu(\tau),\tau)$ is given (i.e., the speed of the rod is measured as it passes the Eulerian coordinate), then $\mu(\tau)$ is determined by a nonlinear second order ordinary differential equation. An integral for the first order ordinary differential equation, $F(\hat{\tau},\tau) = 0$,will be obtained, thereby reducing the determination of $\hat{\tau}(\tau)$ to an algebraic problem. The second order equation for $\mu(\tau)$ will be solved analytically for μ.

Let the fixed laboratory coordinate be at $\mu(0) = \mu_0$. If the end $\mu = 0$ is accelerated at t = 0, this information will arrive at $\mu(0)$ at time $\tau = \mu_0$. At the space-time coordinate $(\mu,\tau) = (\mu_0,\mu_0)$, "initial" conditions on the $(\mu(\tau),\tau)$ path corresponding to the fixed laboratory coordinate are given by

$$\begin{aligned} \hat{\tau}(\mu_0) &= 0 \\ \mu(\mu_0) &= \mu_0 \\ \frac{d\mu}{d\tau}(\mu_0,\mu_0) &= 0 \end{aligned} \tag{28}$$

The third relation follows from eq(25) with $w(\mu,\tau) = 0$ on the $C^{+}(0,0)$ characteristic through $(\mu,\tau) = (\mu_0,\mu_0)$

<u>Case (i)</u>: Having measured $\mu(\tau)$ at the fixed laboratory coordinate it is necessary to determine $\hat{\tau}(\tau)$ in order to determine $\sqrt{\frac{ds}{d\xi}(0,\hat{\tau})}$ from eq.(26). Differentiating eq.(25) with respect to time and using eq.(27) yields

$$\frac{d^2\mu}{d\tau^2} = \frac{d\mu}{d\tau}\left\{\frac{\frac{dw(0,\hat{\tau})}{d\hat{\tau}}}{w(0,\hat{\tau})}\left[1 - \left(\frac{\tau - \hat{\tau}}{\mu}\right)\frac{d\mu}{d\tau}\right]\right\}\frac{d\hat{\tau}(\tau)}{d\tau} \tag{29}$$

Since $w(0,\tau)$ is a known function, eq.(29) is a first order ordinary differential equation for $\hat{\tau}(\tau)$. The initial data needed to determine $\hat{\tau}(\tau)$ is given by the first line of eq.(28). Eq.(28) may be written

$$- \mu\frac{d}{d\tau}\left(\frac{1}{d\mu/d\tau}\right) = - \frac{d}{d\tau}\left[\left(\frac{\mu}{d\mu/d\tau}\right) - \tau\right] = \frac{dw(0,\hat{\tau})/d\hat{\tau}}{w(0,\hat{\tau})}\left\{\left[\left(\frac{\mu}{d\mu/d\tau}\right) - \tau\right] + \hat{\tau}(\tau)\right\}\frac{d\hat{\tau}}{d\tau}$$

A combination of terms forms an exact derivative

$$- \frac{d}{d\hat{\tau}}\left\{w(0,\hat{\tau})\left[\left(\frac{\mu}{d\mu/d\tau}\right) - \tau\right]\right\} = \frac{dw(0,\hat{\tau})}{d\hat{\tau}}\hat{\tau}(\tau)$$

After an integration by parts, an algebraic relation between τ and $\hat{\tau}(\tau)$ results.

$$F(\hat{\tau}(\tau),\tau) \equiv \left[\mu_0 - \int_0^{\hat{\tau}(\tau)} w(0,\hat{\tau})d\hat{\tau}\right] + \left[\left(\frac{\mu}{d\mu/d\tau}\right) - (\tau - \hat{\tau}(\tau))\right]w(0,\hat{\tau}(\tau)) = 0 \tag{29a}$$

In obtaining eq.(29a) the limit as $\tau \rightarrow \mu_0+$ of the ratio of $d\mu/d\tau$ to w implied by eq.(25) has been used.

For illustrative purposes, assume $w(0,\tau) = \tau\alpha$ for $\tau>0$ and $\mu(\tau) = \mu_0 - (\tau - \mu_0)^2 k$ for $\tau - \mu_0 > 0$ with α and k nonzero constants. From eq.(25) $\alpha k>0$. If the velocity at $\mu = 0$ is initially positive (negative) the first nonzero velocity of the bar at the fixed laboratory coordinate is positive (negative), so the Lagrangian mass coordinate at this diagnostic position decreases (increases).

$$F(\hat{\tau},\tau) = \mu_0 - \frac{\alpha\hat{\tau}}{2}\left[(\mu_0 + \tau) + \frac{\mu_0}{(\tau - \mu_0)k}\right] + \frac{\alpha\hat{\tau}^2}{2} = 0$$

$$\hat{\tau}(\tau) - \mu_0 = \left[\frac{\hat{\tau}^2 + \frac{2\mu_0}{\alpha}(1 - \alpha\hat{\tau})}{2\hat{\tau}}\right]\left\{1 - \sqrt{1 - \frac{4\frac{\mu_0}{\alpha}\hat{\tau}^2}{\left[\hat{\tau}^2 + \frac{2\mu_0}{\alpha}(1 - \alpha\hat{\tau})\right]^2}}\right\}$$

$$\hat{\tau}(\tau) = \frac{\left[(\mu_0 + \tau) + \frac{\mu_0}{(\tau - \mu_0)k}\right]}{2}\left\{1 - \sqrt{1 - \frac{\frac{8\mu_0}{\alpha}}{\left[(\mu_0 + \tau) + \frac{\mu_0}{(\tau - \mu_0)k}\right]^2}}\right\}$$

Case (ii): Having measured the displacement velocity with which portions of the bar pass the fixed laboratory coordinate, $w(\mu(\tau),\tau)$ is known whatever $\mu(\tau)$ may be. Consequently, $\hat{\tau}(\tau)$ is known, and it is necessary to determine $\mu(\tau)$ in order to determine $\sqrt{\frac{ds}{d\xi}(0,\hat{\tau})}$ from eq.(26). For this purpose it is convenient to define a new independent variable

$$\lambda(\tau) = \int_{\mu_0}^{\tau} w(\mu(\tilde{\tau}),\tilde{\tau})d\tilde{\tau}$$

(λ is not a displacement corresponding to a Lagrangian coordinate, μ, since time derivatives of λ are not time derivatives of w with μ held fixed.) In terms of λ eq.(25) becomes

$$\frac{d\mu}{d\lambda} = -\frac{1}{1 - \xi} \tag{30}$$

Differentiating eq.(30) and using eq.(7) to express derivatives of ξ in terms of derivatives of w yields

$$\frac{d^2\mu}{d\lambda^2} = -\left(\frac{d\mu}{d\lambda}\right)^2\left(\frac{\tau - \hat{\tau}}{\mu}\right)\left(\frac{dw}{d\lambda}\right)$$

which may be written

$$\frac{d}{d\lambda}\left(\frac{d\ln\mu}{d\lambda}\right)^{-1} = 1 + (\tau - \hat{\tau})\left(\frac{dw}{d\lambda}\right) \tag{31}$$

Two integrations of eq.(31) give μ as a function of λ.

$$\mu = \mu_0 \exp\left\{ - \int_0^{\lambda} \frac{d\lambda'}{\mu_0 - \int_0^{\lambda'} \left[1 + (\tau - \hat{\tau})\left(\frac{dw}{d\lambda''}\right)\right] d\lambda''} \right\}$$

Using the definition of λ yields the needed expression for $\mu(\tau)$

$$\mu(\tau) = \mu_0 \exp\left\{ - \int_{\mu_0}^{\tau} \frac{w d\tau'}{\mu_0 - \int_{\mu_0}^{\tau'} \left[1 + \frac{(\tau - \hat{\tau})}{w}\left(\frac{dw}{d\tau''}\right)\right] w d\tau''} \right\} \tag{32}$$

VIII-Summary and Discussion:

In this paper a new iterative procedure is proposed for determining elastic stress-strain relations for heterogeneous uniform materials (i.e., for those materials whose local stress is not an explicit function of location). The derivation of the proceedure was motivated by resonant bar and nonlinear wave propagation experiments at Los Alamos National Laboratory and assumes that for the data used to determine the state relation no shocks are generated.

For temporally dependent boundary data from cylindrically symmetric samples a "shooting" method is derived in which successive approximations are made to the strain-dependent information paths of the sample. For weak disturbances the first approximation should be very accurate. The method admits the possibility of state relations that are not analytic functions of strain. The method may be readily modified to determine the state relation for samples for which data is provided at a fixed Lagrangian coordinate of the sample (e.g., if an imbedded diagnostic is employed). For displacement velocity data provided at a fixed mass coordinate, the velocity is the same on all points of an early time (i.e., region II of Fig.1) signal path moving from the driven end of the bar. The "mass-time" path between the driven end and the mass coordinate at which data is provided is a straight line. Therefore, the method for obtaining the First Approximate State Relation for boundary data provides the exact state relation when interior data is provided.

While the shooting method can, in principle, be applied to any range of the stress, it should find its greatest applicability in the determination of the state relation for weaker strains. In this range, only the earlier motions of ends of the sample are needed. The information paths determining these motions are straight lines over the vast majority of their trajectory in the mass-time coodinate system (i.e., almost all of the path is in region II). As a result, the FAStR approximation to their true slope at the driven end will be very accurate. If improvements on the FAStR are desired, the extent of region III requiring further analysis will be small. As the range of strain is extended, greater portions of the information paths will be in increasingly larger portions of region III. The accuracy of the initial path direction at the driven end of the sample will diminish, and the difficulty in analysis of curvilinear portions of the information paths will increase.

If appropriate data is provided at a fixed laboratory coordinate (i.e., or the speed of the bar or a time varying Lagrangian coordinate of the bar) a variation of the procedure was shown to yield an "exact" state relation without iteration. The necessary data is not yet available from the Los Alamos experiments. It is not known to the author whether data of this type is available from other laboratories.

The utility of the methods derived in this paper will depend on the accuracy of the data provided. If boundary data is used to implement the procedure, a direct (e.g., optical) diagnostic for velocities should be used whenever possible. Indirect measurements of velocity at an end of the bar may compromise the accuracy of the procedures. For example, if the driving velocity is provided by an ocsillating crystal, a velocity at a free end of a load on the crystal may be measured with an accelerometer. Modelling the load (i.e., crystal plus appended masses) as a single uniform elastic medium, assuming the stress is continuous at the end connected to the bar and vanishes at the free "back" end yields a relation between the velocity measured at the back end, v(back,t), and that at the interface (i.e., the driven end) of the bar and crystal:

$$v(\mathrm{back},t) = v(0,t) + \int_0^{\sigma(0,t)} \left[\left(\sqrt{\rho^*}\, c(\tilde{\sigma}) \right)^{-1} \right]_{\mathrm{load}} d\tilde{\sigma}$$

v(0,t) and $\sigma(0,t)$ are the speed and stress of the interface (i.e., at m = 0 of the previous sections), ρ^* is the unstressed mass density and $c(\sigma)$ is the stress dependent sound speed of the indicated medium. (For simplicity of exposition, the signal-time delay is omitted, so the displayed time, t, in the above equation refers to the same information, or characteristic, path.) The essential point is that the accelerometer on the back end does not measure the velocity of the interface between the driver and the bar. However, if the integrand is small enough, it may provide a very good approximation for that velocity. In that event, the measurement may suffice for obtaining good approximate state relations using the proposed methodology. In general, this requires that the density and signal speed in the load are large compared to those of the sample. For materials with these properties, a strain applied at a boundary surface is rapidly released and propagates primarily as kinetic energy.

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